Classical and Advanced Kepler Algorithms

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Abstract

Space Surveillance is a critical and computationally intensive function of Space Situation Awareness (SSA), if updating position database of 100,000 or more objects is required. When predicting a few days of conjunction and collision assessments, debris removal scheduling, or other SSA applications, efficient trajectory predictions or position updates is key. Intelligence, reconnaissance, environment and command & control are other SSA functions outside the bounds of Astrodynamics.

DerAstrodynamics presents the classical and advanced Kepler algorithms for analytic trajectory prediction in this paper. The superior multi-revolution Lambert algorithm for targeting and the new range-solving Gauss/Laplace algorithms for angles-only are also available online. These three analytic SSA algorithms are fast, accurate and robust. In general, all classical Kepler, Lambert, and Gauss/Laplace angles-only algorithms produce Keplerian (2-body) solutions, while our advanced algorithms are orders of magnitudes more accurate by including important perturbations of 4x4 Geopotentials, Sun, Moon and drag.

Classical Kepler algorithms [1 to 19], to name a few, have been described in numerous textbooks and papers over 400 years. A classical Kepler algorithm, which is the analytic prediction algorithm of 2-body orbital mechanics, is not an accurate trajectory predictor for SSA. An advanced Kepler algorithm improves accuracy by adding perturbations analytically without much loss of computational efficiency. However, without the "inaccurate" 2-body solutions, all other accurate advanced prediction, targeting and angles-only algorithms would not have existed. The SGP4, Vinti, Lambert, and Gauss/Laplace angles-only algorithms all require 2-body solutions for initial guesses. In general, if an advanced analytic Astrodynamics algorithm fails to solve the Kepler equation, then it breaks down as well.

For various reasons, computer programs that subject classical Kepler algorithms to extensive testing on CPU timing, accuracy, robustness have not been available until now (**Kepler_Test**: http://derastrodynamics.com/index.php?main_page=index&cPath=101_1_10) *DerAstrodynamics* presents source code of two driver computer programs (**Kepler_Test1** and **Kepler_Test2**), three classical Kepler algorithms (kepler1a, kepler1b, kepler2) and six numerical integrators for extensive testing of Kepler algorithms. Data of over 100,000 initial position and velocity vectors of cataloged objects in all orbit regimes are also provided. The reader can replace the kepler1a or kepler1b algorithm [18] by his or her preferred Kepler algorithm. The kepler2 algorithm has been tested extensively (over five billion arbitrary test cases) without a single failure.

DerAstrodynamics also presents an advanced Kepler orbit computation tool, **iSGP**:
http://derastrodynamics.com/index.php?main_page=product_info&cPath=101_1_11&products_id=100029

Introduction

An analytic Kepler algorithm provides analytic initial predictions (position and velocity vectors) to construct pre-conjunctions. Numerical integration is invoked only for close conjunctions. If the pre-conjunctions are inaccurate, all close conjunctions are meaningless leading to numerous false alarms. This study and other published papers assert that the SGP4 algorithm has robustness and accuracy prediction difficulties leading to meaningless results. The **iSGP** tool, which is created to substantiate these assertions, used the new advanced Kepler algorithm, Kepler+, to eliminate these difficulties analytically. If accurate data from the SP Catalog (instead of the TLE Space Catalog) is available, then close conjunctions can be predicted efficiently with Kepler+ on a laptop computer. However, if computing costs and time are unlimited, conjunctions by SP data and numerical integration are preferred, and pre-conjunctions by analytic algorithms are not needed.

Directly quoting from R. H. Battin [1], "Algorithms for the solution of Kepler equation abound." References 1 to 19 deal with algorithms that solve the Kepler equation by different initial guesses and/or various types of iterative and non-iterative methods. In particular, References 9 and 10 develop methods that solve the Kepler equation by direct, non-iterative methods. Reference 11 utilizes the combination of direct and iterative methods. Reference 12 presents an integrative method using homotopy theory. Reference 13 is a book that summarizes over three centuries of methods to solve the Kepler equation. References 14 to 18 present some of the best universal variable formulations of the Kepler equation. Reference 19 compares the Kepler and Lambert algorithms, and picked the winners based on a small number of test cases. Even with sophisticated methods for initial guesses, very few of the analytic, graphical, direct, integrative and iterative methods developed over 400 years can guarantee convergence for all trajectories. Most of these references are available online at:
http://derastrodynamics.com/index.php?main_page=product_info&cPath=101_119_121&products_id=100035.

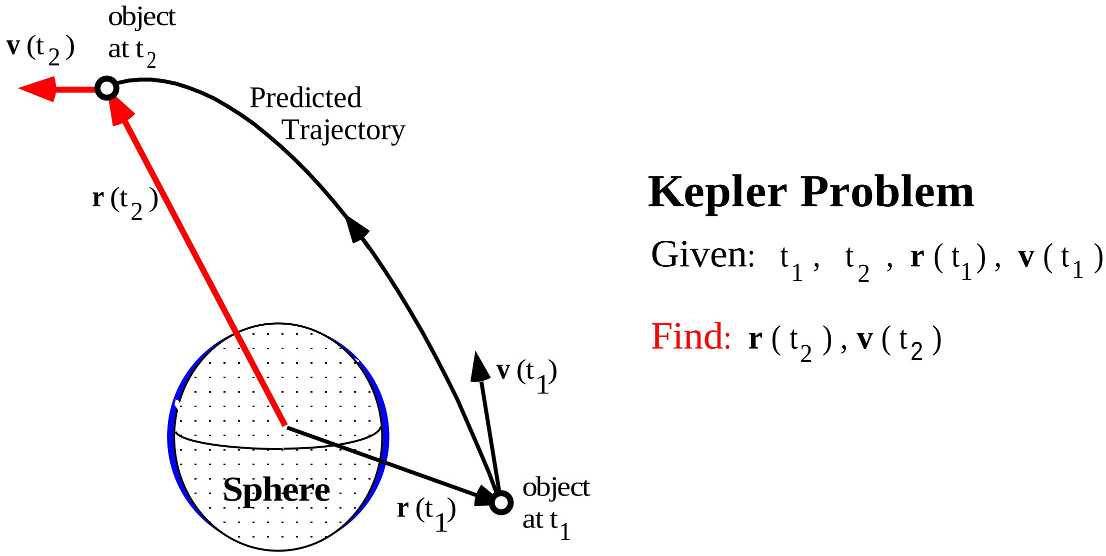

Figure 1. The Kepler problem

As illustrated in Figure 1, the Kepler problem is an initial value problem of solving the equations of motion

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu}{r^3}\,\mathbf{r} \qquad (1)$$

to find the position and velocity vectors $\mathbf{r}(t_2)$ and $\mathbf{v}(t_2)$ at any given time t_2; given also the initial position and velocity vectors $\mathbf{r}(t_1)$, $\mathbf{v}(t_1)$ and the initial time t_1, and μ is the gravitational constant for the relevant spherical body. Equation (1) looks simple, but is hard to develop a robust algorithm that can "always" produce the correct solution for any conic trajectory. Under what condition(s) a Kepler algorithm breaks down? Nobody knows. A Kepler algorithm may fail due to any or all of the following: 1. the iterative method (Newton, Halley, Laguerre, . . ,), 2. the initial guess for the iterative method and 3. the correction formula after each iteration. Klumpp [19] addressed part of these test metrics using 115 tough test cases, but that is just the first step.

The formulation of a Kepler algorithm or equation in either classical orbital elements or universal variables is not the cause of failure. The classical theory ends with solving for the solution of one unknown, the eccentric anomaly E, in the Kepler equation:

$$F(E) = E - e\sin E - M = 0 \qquad (2)$$

where the mean anomaly M and the eccentricity e can be computed from the given $\mathbf{r}(t_1)$ and $\mathbf{v}(t_1)$. The Kepler equation formulated by the Universal variable, x, can be expressed as:

$$F(x) = r_i U_1 + \sigma_i U_2 + U_3 - \sqrt{\mu}\,(t - t_i) \qquad (3)$$

where the derivation of equations (2) and (3) can be found in many papers and text books [1, 18]. After E or x is found, the required $\mathbf{r}(t_2)$ and $\mathbf{v}(t_2)$ at t_2 can be evaluated.

Equations (2) and (3) are analytic but not in closed form, the Laguerre iterative method suggested by Pressing and Conway [5, 18] gives the iterative formula

$$x_{i+1} = x_i - \frac{5\,F(x_i)}{F'(x_i) + \frac{F'(x_i)}{|F'(x_i)|}\sqrt{16\,\left(F'(x_i)\right)^2 - 20\,F(x_i)\,F''(x_i)}} \qquad \text{for } i = 0, 1, 2, .. \qquad (4)$$

where the first and second derivatives of $F(x)$ are respectively $F'(x)$ and $F''(x)$. The kepler1a and kepler1b algorithms [18] provide (1) simple initial guess for circle and ellipse $x_i = \alpha\sqrt{\mu}\,(t - t_i)$ and for parabola and hyperbola $x_i = \sqrt{\mu}\,(t - t_i)/(2r_i)$ where $\alpha = 1/a$ and (2) the upper and lower bounds provide $x_i = 0.5\sqrt{\mu}\,(t - t_i)\left[1/a(1-e) + 1/a(1-e)\right]$, where a and e are the semi-major axis and eccentricity. However, these initial guesses can cause the Kepler algorithm to fail as shown in Examples 1 and 2 of this paper. These two examples are intended to demonstrate the overlooked difficulties of solving the Kepler equations (2) and (3), and should not be assumed or perceived to discredit the excellent works of Professors Prussing and Conway. If equation (4) is replaced by:

$$x_{i+1} = x_i - \frac{n\,F(x_i)}{F'(x_i) \pm \frac{F'(x_i)}{|F'(x_i)|}\sqrt{(n-1)^2\left(F'(x_i)\right)^2 - n\,(n-1)\,F(x_i)\,F''(x_i)}} \qquad \text{for } i = 1, 2, .. \qquad (5)$$

and n is allowed to vary whenever the convergence condition $\left| x_{i+1} - x_i \right| \leq 1.0^{-12}$ cannot be satisfied after 10 iterations of any n, starting with n = 2.

The iterative method using (5) is "almost" bullet proof. The accuracy of a classical Kepler algorithm can be compared with the accurate numerical integrated solutions of equation (1). The six integrators of the **Kepler_Test** package allow users to see the variations of the integrator step-size and accuracy of the numerical solutions. The initial guess and the correction formula can be improved after billions of arbitrary test cases.

The choice of reference numerical integrated solutions to compare with advanced Kepler algorithms is not straightforward. For SSA initial orbit determination and position updates, the predicted magnitude of position differences between analytic and reference solutions should be less than a few kilometers per day. Many papers [20, 21] on numerical integration of trajectories compare numerical programs (GEODYN, GTDS, . . ,) Geopotential models (EGM, WGS) and other perturbations indicate that for practical SSA purposes the reference numerical solutions of WGS84 12x12, Sun, Moon, and drag are sufficient. Solar radiation pressure and other perturbations are excluded without much loss of accuracy. Many of these reference numerical solutions have been compared and matched within 100 meters of position per day with data from the Astrodynamics Work Station (ASW).

The survey paper by Vetter [22] shows that almost all US Government research funding on prediction algorithms in the last 50 years was spent on numerical integration computer programs. Rapid advances of super computers allow numerical integration to compute trajectories of a few hundred or thousand objects within centimeter accuracy. However, when 100,000 objects and billions of trajectories are predicted for just one day, numerical integration will be hard to keep up, unless unlimited computing power and memory resources are available. For conjunction assessments, brute-force numerical solutions are not just mostly unnecessary, they cannot provide the "feel" or understanding to allow efficient development of conjunction algorithms. Worst, some computer programs built for extremely high accuracy can be difficult switching to lower accuracy for pre-conjunction assessments.

Space debris is posting financial and safety concerns for operating satellites. The Iridium and Cosmos satellite collision on 10 Feb 2009, the near-miss of the NASA Fermi and Cosmos1805 on 3 April 2012, the Chinese Fengyun Debris and Russian Blits collision on 22 Jan 2013, and many small collisions are just reminders that fast and accurate analytic prediction algorithms are needed urgently. Other functions of SSA, such as multi-sensor multi-object Uncorrelated Target (UCT) cataloging, scheduling of space debris removal, just to mention a few, also need these analytic algorithms.

The advanced Kepler algorithms: SGP4, Vinti and Kepler+ will be described. First, the prediction performance parameters of these analytic advanced algorithms are compared and summarized in the Tables 1, 2 and 3 in the next section using thousands of cataloged objects. Many days of Two Line Element (TLE) files from year 2001 to 2013 were arbitrary chosen, and obsolete TLE data was removed. Since the results are similar, only three TLE files are sufficient to show typical prediction performances. Initial errors of the TLE state vectors are neglected, but must be accounted for in accurate or close conjunction assessments.

Comparison of Advanced Kepler Algorithms

Algorithm	CPU timing micro-seconds per trajectory			Perturbations	Singularity
	6 Sep 2007 11140 objects	25 Oct 2009 14327 objects	5 Jan 2011 14638 objects		
SGP4 (analytic)	18.2	17.4	17.4	J_2, J_3, J_4 Sun, Moon, Drag	Yes
Vinti (analytic)	14.0	14.2	13.9	J_2, J_3, J_4	No
Kepler+ (analytic)	8,400.	8,000.	7,800.	J_2, J_3, J_4, J_{22}, J_{31}, J_{32}, J_{33}, Sun, Moon, Drag	No
Numerical Integration (4x4, 200s)	127,000.	127,000.	127,000.	WGS84 4x4 Sun, Moon, Drag	No

Table 1. CPU timing, perturbations and singularity comparison of advanced prediction algorithms for one-day prediction using thousands of cataloged objects

Algorithm (Reference = WGS84 12x12, Sun, Moon, Drag)	Robustness, mean and standard deviation (std) for one-day prediction					
	6 Sep 2007 total # of objs = 11140		25 Oct 2009 total # of objs = 14327		5 Jan 2011 total # of objs = 14638	
	# of objects with position difference > 5 km	mean / std (km)	# of objects with position difference > 5 km	mean / std (km)	# of objects with position difference > 5 km	mean / std (km)
SGP4 (analytic)	7031	52. / 425.	9026	50. / 241.	9289	46. / 195.
Vinti (analytic)	4238	6.1 / 27.1	5596	7.1 / 90.0	5648	6.1 / 11.8
Kepler+ (analytic)	1110	2.3 / 2.0	2002	2.7 / 2.4	1942	2.6 / 2.5
Numerical Integration (4x4, 200s)	133	1.5 / 1.4	217	1.7 / 1.4	176	1.6 / 1.3

Table 2. Robustness, mean and standard deviation comparison of advanced prediction algorithms for one-day prediction using thousands of cataloged objects

Algorithm (Reference = WGS84 12x12, Sun, Moon, Drag)	Robustness and max error with respect to Reference for one-day prediction					
	6 Sep 2007 total # of objs = 11140		25 Oct 2009 total # of objs = 14327		5 Jan 2011 total # of objs = 14638	
	# of objects with position difference > 10 km	max error (km)	# of objects with position difference > 10 km	max error (km)	# of objects with position difference > 10 km	max error (km)
SGP4 (analytic)	4760	37925.	6243	13138.	6161	6025.
Vinti (analytic)	2174	2443.	2875	10212.	3031	709.
Kepler+ (analytic)	34	28.	105	62.	85	131.
Numerical Integration (4x4, 200s)	7	20.	11	25.	6	15.

Table 3. Robustness and maximum errors from the reference comparison of advanced prediction algorithms for one-day prediction using thousands of cataloged objects

General Comments
Tables 1, 2 and 3 compare the prediction performances of advanced Astrodynamics prediction algorithms in terms of CPU timing, accuracy, robustness, mean, standard deviation and maximum errors with respect to a reference using over thousands of cataloged objects (three arbitrary days of TLE files). Reference solutions should technically be accurate measurement data, which are usually not available to the public. For the sake of comparison, reference solutions are those of numerical integration with WGS84 Geopotential 12x12, Sun, Moon and drag. Higher-order Geopotential models (WGS, EGM, 96x96, . .), solar radiation pressure, and others introduce errors less than 100 meters per day, and therefore the above prediction performance parameters are still valid. Since trajectories of over 11,000 objects at different date are used, Tables 1 to 3 are sufficient to provide a reasonable guideline. Note that the Kepler algorithm is imbedded in the three advanced analytic algorithms. Test drivers similar to **Kepler_Test1** and **Kepler_Test2**, which the reader can easily add the drag parameters, BSTARs, to the object_eci_input files, are available upon request from *DerAstrodynamics*.

From Table 1, the analytic Kepler+ algorithm, which includes "almost all" of the 4x4 Geopotentials, is about 15 times faster than numerical integration with the same perturbations (faster with optimization in **iSGP**). Speed improvement is a trade with accuracy. Robustness is measured by the numbers of objects that have respectively position magnitude differences greater than 5 km and 10 km with respect to the reference solutions in Tables 2 and 3. Accuracy is measured by the mean, standard deviation and maximum errors. Note that the SGP4 results are computed by a version similar to a 2006 official version.

Particular Comments

CPU timing numbers in Table 1 are in micro-seconds. SGP4 and Vinti are fast, at about 18 and 14 micro-seconds respectively on a 2012 personal computer. The Kepler+ algorithm and comparable numerical integration average respectively 8,000 and 127,000 micro-seconds per trajectory prediction. Optimization improves the Kepler+ average speed to about 500 micro-seconds executed in the new tool, **iSGP**. The SGP4 algorithm, which has the most and largest differences with respect to the references, is the only predictor that has singularity problems. Singularity leads to unpredictable large errors for SGP4 as shown in Tables 2 and 3, even after outliers were removed. Singularity is not desirable for conjunction assessments.

Tables 2 and 3 illustrate the performance parameters of robustness, mean, standard deviation and maximum errors of position magnitudes with respect to the reference solutions. There are obsolete data in each TLE file of the Space Catalog. On 6 September 2007, the Space Catalog has 12,049 objects, but only 11,140 are used. The outliers, which are removed, are dated either older than 10 days or "later" than the date the catalog was published. The serious readers can use the TLEs of any other day to verify the SGP4 predictions against the reference numerical solutions that include the WGS84 Geopotential 12x12 model, Sun, Moon and atmospheric drag. An interesting accuracy comparison, which is not included in this paper, is to use the updated TLEs of the next day as another data point to compare the SGP4 predicted solutions. Readers will be surprised how good or poor SGP4 predictions compared with its own future data using all updated objects from one day to the next.

The numbers in the column of Table 2 under > 5 km or Table 3 under > 10 km illustrate the number of objects with position magnitude difference between the stated algorithms and the reference solutions are greater than 5 or 10 km. For example: in the 2007 test case SGP4 has 7,031 out of 11,140 objects with position magnitudes greater than 5 km with respect to the reference in Table 2. For the same test case, the mean and standard deviation that indicate uncertainties and bounds are respectively 52 and 425 km. Trajectory prediction with speed is desirable but not about 50% or more of the cataloged objects are so inaccurate and uncertain for predicting just one-day ahead. Conjunction assessments using SGP4 for pre-conjunction with subsequent minimum range of less than one kilometer by some Internet websites and published papers are operating on the strategy of "Blind Leading The Blind (BLTB)."

SGP4 is a good predictor for pointing predictions of radar and optical sensor on well tracked objects, because a radar fence or an optical sensor field-of-view can accommodate easily 5 or 10 km of errors in pointing. However, 5, 10, or even over 100 km of one-day prediction errors cannot justify those 10 to 100 meter conjunctions published daily on those BLTB websites. In other words, using any algorithm with large prediction errors and uncertainties to compute conjunction and collision assessments leads to not just large number of false alarms, but anxieties and unnecessary costs of executing collision avoidance. The 2009 collision of Iridium33 and Cosmos2251 and other collisions were not predicted ahead by those websites, and excuses of inaccurate initial data, funny conjunction probability or unlucky maneuver cannot cover up their inability to predict accurately using the fundamentally flawed SGP4 algorithm for conjunction assessments. Again, the performance data in Tables 1 to 3 can be verified easily by comparing the analytic solutions against the reference numerical solutions using an arbitrary TLE file.

SGP4 Algorithm

The research data of Tables 1 to 3 is to illustrate the performance of analytic prediction algorithms for SSA. They should not be assumed or perceived to discredit SGP4 on its theory, formulation and the excellent contributions of friends who coded the SGP4 algorithms.

The Simplified General Perturbations (SGP) and the more accurate version SGP4 have been assumed as the only analytic prediction algorithms for Earth satellites and debris objects in Astrodynamics since the 1960s. The following may be some of the reasons:

1. It is hard to justify a wrong decision had been made by many knowledgeable people over 50 years ago. In addition, a lot of resources and effort have already been spent.
2. The authority is led to believe that there is no better analytic predictor to replace SGP4. Ignoring politics, very few have the technical know-how to understand the SGP4 difficulties, let alone presenting an alternative predictor. Besides, there are also many "improved" SGP4 versions with unofficial and official Band-Aid fixes.
3. The fact that some important satellites often need costly reacquisitions, not because they suddenly disappeared or crashed, indicates hundreds of kilometers of SGP4 predicted pointing errors. These are ignored easily, when costs are not important.
4. Most SGP4 performance reports show great accuracy performance using only a few "SGP4 friendly" objects. However, SSA applications require processing over tens of thousands of objects in the Space Catalog. The performance metrics of Tables 1 to 3 would have shown that the emperor has no clothes, but few wants to stop the play.

The SGP4 algorithm was formulated with singularities, which occurs when something is divided by zero or a very small number. Some of the SGP4 difficulties are:

1. Built-in singularities at or near zero and 63 degree inclinations.
2. Built-in singularities at or near zero eccentricities.
3. Kepler convergence failure. When SGP4 calls for an initial guess of the predicted solution, the returned solution from Kepler Equation (2) or (3) is wrong.
4. High eccentricity ($e > 0.3$) can produce unpredictable solutions.
5. Drag calculation can be erroneous when eccentricity is high ($e > 0.1$).
6. The input parameters (mean orbital elements and others) in a TLE file are in single precision and the computations of SGP4 are in double precision. Forward or backward SGP4 prediction from single precision mean orbital elements can produce easily kilometers of errors in one day.
7. Even though the conversion of osculating orbital elements to mean orbital elements computations are in double precision, squeezing the converted double precision mean values to fit the single precision TLE format will not allow SGP4 to re-compute the original osculating orbital elements.

The SGP4 computed results of our 2006 version were verified to match all those of the TRAIL (semi-official) test cases, and those of References 21 and 23. Unfortunately more than 40% of the SGP4 predictions are over 10 km in differences with respect to accurate numerical reference solutions using any TLE file with more than 10,000 objects for just a one-day prediction. Many at NASA [24] are convinced that SGP4 is flawed, but they are uncertain how badly (Table 3). If the objective is catalog maintenance or sensor pointing, then SGP4 is a reasonable tool for most predictions. For accurate position updates and conjunctions, SGP4 needs more than a few Band-Aid improvements. Our innovative sgp algorithm of **iSGP** can "sense" the SGP4 difficulties, and then replace all unreasonable SGP4 solutions.

Raw radar observations and tracking data can be easily transformed to osculating position and velocity vectors. However, SGP4 requires the input to be the mean orbital elements or TLEs as in a TLE file. This requirement presents difficulties for the SGP4 algorithm to predict unknown or uncataloged objects that have no TLEs at the time of detection, such as a missile or an uncorrelated target (UCT). When TLE input is not available, the conversion of osculating elements to mean elements (osc2mean or o2m) needs be performed [25, 26] before SGP4 can be used for predictions. It is difficult to build a robust conversion algorithm when Keplerian orbital elements must be used. This simple conversion appears be painful for most engineers, the o2m algorithm of **iSGP** alleviates this difficulty.

The original SGP and SGP4 algorithms were developed in the 1960s by Kozai, Hilton, Brouwer, Lane and Cranford, [27 to 30]. The formulation began with the Hamilton-Jocobi equations of motion. Instead of analytic integration, "averaging" technique was used, resulting in the "mean" orbital elements as input for SGP4. The output of SGP4 is the (osculating or instantaneous) position and velocity vectors. Note that, the input and output of Vinti, Kepler+ and numerical integration are osculating position and velocity vectors. Assuming no singularity, the SGP4 algorithm provides a reasonable solution at least a few orders of magnitudes more accurate than the comparable 2-body solution for most objects other than those at GEO. Unfortunately, SGP4 is, in general, the least accurate advanced Kepler algorithm as shown by the results of Tables 1 to 3 and A1 to A3.

The original version of SGP4 was documented by Hoots [31] and the computer source code is also publicly available [31, 32]. An official version became in use around 1990. Numerous versions of SGP4 were developed and improved by users as described by Vallado [23], but they cannot eliminate all SGP4 difficulties under any condition. Typical differences in initial position magnitudes between the publicly available version [31] and an official version are about a few kilometers. The prediction performance parameters of the unofficial and official versions are similar as those in Tables 1 to 3.

Mean orbital elements are updated continuously, but a new TLE file is published every day. Catalog maintenance requires a lot of resources. Comparing the epochs of the objects in consecutive days, about 40% of the objects in the whole catalog can be a few days old. Researchers from the US, Germany, Australia and Canada [33 to 36] indicate that the initial TLE bias errors compared with measurement data ranges from 100 meters to 2 km, and SGP4 prediction error is approximately 0.1 to 3.0 km per day using small samples of SGP4 friendly objects. However, large samples or all objects in a TLE file must be considered for SSA applications, and prediction errors can be 5 km per day or much more for nearly 50% of the cataloged objects as shown in Tables 1 to 3. Directly quoting from Levit [33], " predictions based on TLEs using the associated analytic propagator (SGP4) are not sufficiently accurate to warrant maneuvering to avoid potential collisions: resulting in an unacceptably large number of potential collisions per space object, each of which has very low probability" By ignoring this assertion and our research data, all SGP4 initiated conjunction assessments are meaningless and hard to believe.

iSGP Algorithms

The **iSGP** package contains three algorithms to verify and replace the solutions of SGP4 if necessary, and to convert osculating position and velocity vectors to TLEs if needed:

1. **ref** includes the reference numerical integrated solutions that accuracy of SGP4 and Kepler+ solutions can be verified.
2. **sgp** replaces all erroneous SGP4 solutions due to singularities, Kepler convergence failure and other difficulties with reasonable Kepler+ solutions.
3. **o2m** converts osculating position and velocity vectors to the six mean orbital elements as in a TLE file.

The primary objective of the **iSGP** algorithms is to eliminate all "big errors" or large position differences between SGP4 and reference solutions (from about 50 to 1,000s of km in position magnitude). The **sgp** algorithm of **iSGP** can distinguish instantly a big erroneous solution of SGP4 without numerical integration. For small errors (from about 5 to 50 km), the solutions of the **sgp** algorithm are often better, but not perfect. If any SGP4 solution is considered inaccurate, then the reliable Kepler+ solution is computed and the SGP4 solution is replaced.

The serious reader can compare his/her numerical integrated solutions against the reference solutions of the **ref** algorithm, which includes perturbations of WGS84 12x12 Geopotentials, Sun, Moon, and drag. Note that very few GEO solutions are replaced, because their 2-body solutions are reasonably accurate for initial orbit determination (IOD). However, GEO objects constitute only 5% of a 20,000 object Space Catalog. If 2-body solutions are used for the other 95% objects, then IOD results are accurate only for short trajectories and mostly inaccurate for a perdition time of one day or longer. Appendix A contains the analysis of accuracy and robustness of Keplerian (2-body) solutions for one day prediction.

Vinti Algorithm

The analytic Vinti algorithm [32] is an advanced Kepler algorithm. The Vinti theory was developed in 1959, and the contentious argument of singularities in methods of general perturbations began about 1959, see our Brouwer vs Vinti article: http://derastrodynamics.com/docs/brouwer_vs_vinti_v1.pdf.

Brouwer insisted that singularities exist in general perturbations in his letter to Vinti on December 23, 1959. Vinti countered that singularities can be eliminated with his elegant formulation. Brouwer won the battle, but the SSA community lost the war. SGP4 has been producing incorrect predictions whenever difficulties are encountered. SGP4 supporters are either afraid to see the truth by not performing a one-day prediction as those of Tables 1 to 3, or they know the truth, but have covered it up for over 50 years. SSA should not allow incorrect position updates of 10 km let alone 1,000 km or more. Professor Vinti would be pleased to see that the singularity problem of SGP4 is fixed once for all. Previous published papers, current and ongoing research on Vinti theory in many countries show that versions of Vinti algorithms exist and are invaluable to missile and satellite predictions and targeting.

Directly quoting from Chapter 5.12 of GTDS [38], "Vinti theory is a general perturbation method. In an approach that is similar to Brouwer, . . . The resulting solution gives the periodic terms correctly to order J_2^2 and the secular terms for the intermediate orbit to arbitrarily high order. . . This method of treating the effects of J_3 eliminates singularities for small eccentricities and for small or 180-degree inclinations, which usually occur in perturbation theories. . ." end quote. In other words, the Goddard Mathematical Theory stated that the Vinti algorithm is theoretically singularity-free. The Vinti theory is documented in Vinti's AIAA book [32] together with a version of source code in both Fortran and C, vinti6. The serious reader can modify the **Kepler_Test1**, and **Kepler_Test2** to deduce the performance parameters of the Vinti algorithm as those in Tables 1 to 3. Note that kepler1 should be replaced by kepler2 for the initial guess to vinti6. Other improvements on vinti6 may be available upon request.

How can Vinti outperform SGP4 for one-day prediction without perturbations due to Sun, Moon, and drag? More than 50% of the objects in the current Space Catalog have perigee altitudes between 600 and 1,600 km, where the acceleration due to drag is minimal for one-day prediction. Gravity is a weak force that takes time to develop. Accelerations due to the Sun and the Moon on all the cataloged objects except those in GEO and above are also small for one-day prediction. The performance parameters in Tables 1 to 3 using large number of objects (over 10,000 per TLE file) are deduced by comparing the solutions of analytic algorithms (SGP4, Vinti and Kepler+) against reference numerical solutions. Longer time prediction results will be included in a future paper. However, if your one-day position predictions are over 10 km in errors for more than 40% of the cataloged objects, your position updates and conjunctions will be worse for longer time predictions. Any website that publishes minimum ranges or miss-distances of a few meters with the use of SGP4 to initiate conjunction assessments appears to be pulling numbers out of thin air. Any Astrodynamics textbook that does not include the Vinti theory has lowered its value below the Goddard Mathematical Theory [38], deprived students to learn the real analytic algorithm [32], and mislead the uninformed to use lesser analytic algorithms that cannot make SSA work.

The Vinti algorithm can outperform SGP4 in general, but not always. The Kepler+ algorithm, which begins with Vinti and includes most of the 4x4 Geopotentials, Sun, Moon and drag, is proposed to be the initial prediction algorithm for SSA position updates, conjunction and collision assessments.

However, before describing the Kepler+ algorithm, there are serious misunderstand and misleading interpretation of Vinti that must be clarified first.

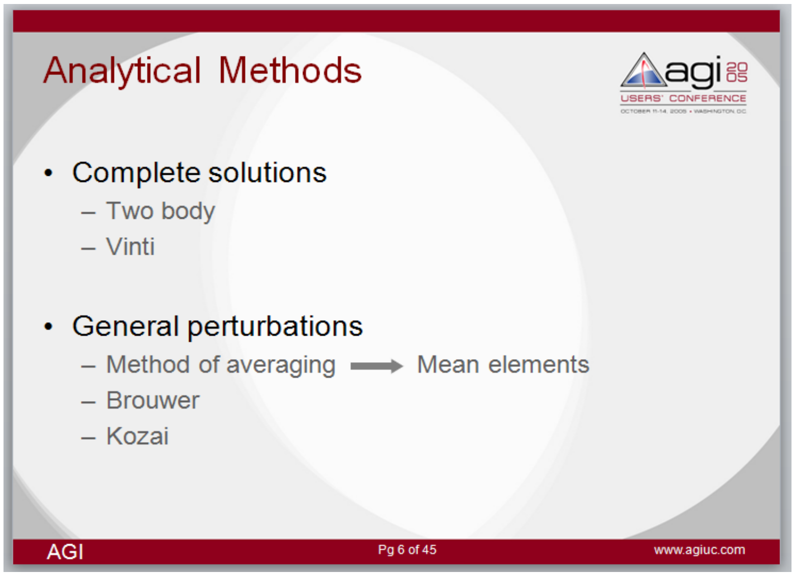

Figure 2. Woodburn of AGI's misleading interpretation of Vinti

The chart in Figure 2 belongs to the presentation for Conjunction Analysis Fundamentals [37] by Woodburn of AGI, and is available online from the AGI website. Either the statement of GTDS [38], "Vinti theory is a general perturbation method", is wrong, or the Woodburn's interpretation of Vinti is wrong. As described in the SGP4 section above, the Brouwer and Kozai theories were used to develop the SGP4 algorithm by the method of averaging and mean elements. Woodburn's lack of understanding of general perturbations would not lead him to study the SSA performance comparisons as illustrated in Tables 1 to 3. However, with so many people like Vetter [22], who have covered-up Vinti for over 50 years, it may be easy to just ignore or exclude Vinti from General Perturbations as shown in Figure 2. Since the Goddard Bible cannot be wrong, it is apparent that Woodburn is wrong.

Woodburn's other misleading comments on Vinti:

1. Is not a solution of the J_2 problem.
2. Requires Lagrangian or Hermitian interpolation.
3. Is not accurate.
4. Needs something more difficult to teach in graduate classes.

Again, the Vinti theory and computer source code presented in the Vinti's 1998 AIAA book [32] show clearly the Vinti algorithm is not what Woodburn stated. The truth is:

1. Vinti theory as stated in GTDS [38] and the publicly available computer source code [32] demonstrated that the Vinti algorithm can solve not just the J_2 problem alone (setting $J_3 = 0$), but includes J_3 and most of J_4 perturbations as well.
2. Vinti does not use Lagrangian or Hermitian interpolation.
3. Vinti is more accurate than SGP4 using SSA performance parameters (Tables 1 to 3).
4. The theories of Vinti, Brouwer and Kozai all begin with the Hamilton-Jacobi equations of motion (the entry ticket to Quantum Mechanics), which are much more difficult to teach than most advanced courses in graduate classes.

The vinti6 algorithm was developed initially by a team of engineers and scientists at the National Security Agency in the early 1970s [39]. They were able to get that version to work for only one test case. The NSA original version (Vinti5) [32] shows that the Vinti theory is not just difficult to teach, it is difficult to code as well. Since the Vinti source code was available in 1998, there was not even one line of code change suggested by anyone to the author! However, a friendly reader from APL indicated the Kepler convergence problem in the kepler1 algorithm, and the kepler2 algorithm in the **Kepler_Test** package should be used: *http://derastrodynamics.com/index.php?main_page=index&cPath=101_1_10*

Woodburn's AGI presentation [37] shows that either he does not understand general perturbations or has ignored the works of GTDS [38], the Vinti's book and computer source code [32], and numerous of papers on Vinti theory in the last 50 years. Worst of all, his J_2 and J_4 propagators are inferior to SGP4. If these are the algorithms Woodburn is using for conjunction assessments, then he will have difficulties develop tools needed to make SSA work as indicated below.

Directly quoting from the Aviation Week & Space Technology/March 19/26, 2012 article [40], " . . . . . In the wake of the collision, Intelsat, SES and Inmarsat formed the Space Data Association (SDA) to develop techniques for merging and disseminating their satellite-control data to improve collective SSA. Later, Eutelsat and about a dozen other operators joined, and the "not-for-profit" organization hired Analytical Graphics Inc. to develop the computer tools needed to make it work. . . . . Iridium Communication Inc. is a member of SDA, but John H. Campbell - the company's vice president for government programs - "Our real concern is debris and avoiding all that, and the LEO environment is a lot more dynamic environment, and as we understand it SDA doesn't really provide any conjunction assessment against 20,000 or so pieces of debris that are in the LEO environment," says Campbell, a retired Air Force lieutenant general. . . . . . . " end quote. However, AGI was hired by SDA. Woodburn of AGI can claim that his conjunction assessment tool works, but it is hard to believe given what was said in Aviation Week [40] and published results of many papers [33 to 36].

In support of General Campbell, Chapter 9 of the National Research Council 2011 report [24] wrote, " . . . While two-line element sets are available to NASA and the public, several problems exist with these data. First, to propagate these data correctly, access to the same orbital model SGP4 used to generate the data is required. . . . . has not released it to the public since 1980. Without the changes made to this model since then, it is possible to have errors on the order of 1,000 km in GEO. Second, while these data are generally good enough for their intended purpose to maintain tracking by the U.S. Space Surveillance Network (SSN), their associated uncertainties . . . contribute to high false-alarm rates and discourage efforts to perform CARA. The large uncertainties are the result of the nature of the SSN tracking that does not take into account maneuvers when computing the TLEs. It is not unusual for it to take a week or more to detect maneuvers and update the associated orbits. It is also common to see objects cross-tagged (switched) within the GEO population, since the noncooperative . . ." end quote.

The TLE data errors, maneuvering and noncooperative objects, cross-tagging GEO objects and lack of covariance data in the TLE file are important problems. However, the 20,000 or so pieces of debris objects in LEO, which contributes to over 80% of the cataloged objects, are in ballistic non-maneuvering orbits. The LEO orbit regime is where collisions will most likely occur if history and statistics are the guides. The prediction errors of the LEO objects must be reduced, and the other 10 to 20% of the cataloged non-LEO objects should be dealt with simultaneously or later.

In summary, if most of your predicted position vectors in pre-conjunction contain errors of 10 km or more after one day, all close conjunctions (by subsequence numerical integration) of less than one kilometer are practically useless, no matter how you spin the results.

Kepler+ Algorithm

The Kepler+ algorithm extends the Vinti algorithm by adding analytically more perturbations so that accuracy is close to those of numerical integration (WGS84 4x4 Geopotentials, Sun, Moon and drag), but much faster than numerical integration. CPU timing results can be easily confirmed by comparing the timing results of the **ref** and **sgp** algorithms of **iSGP** Demo Tool. The Vinti theory was decades ahead of its time. To avoid getting too far ahead, the Kepler+ algorithm will only be described without details.

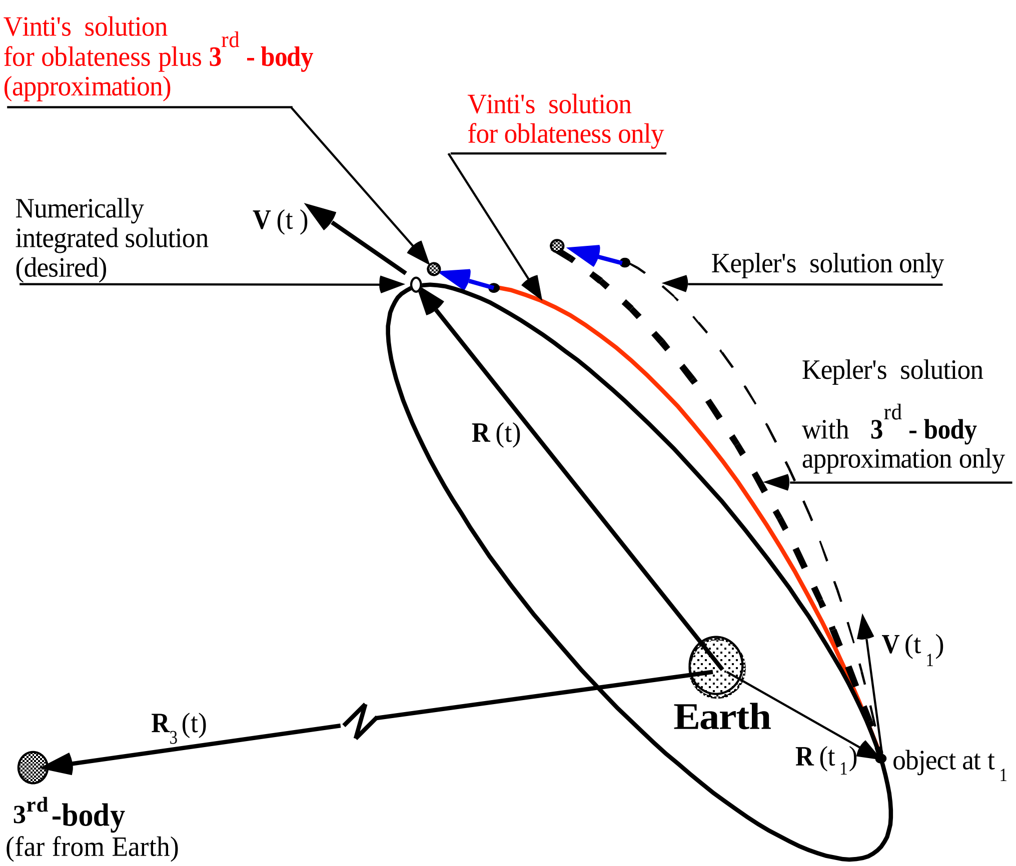

Figure 3. Conceptual addition of 3 rd -body approximation to the Vinti solution

Figure 3 illustrates the conceptual analytic addition of 3^{rd} body gravitational acceleration to the fast, accurate and robust Vinti solution. Using the integrals of the electro-mechanical oscillation due to forced oscillation, an analytic approximation of the 3^{rd} body acceleration can be added to the Vinti solution (blue arrows). This technique of analytic addition of perturbations is similar to that of Reference 41.

Simple analytic approximations allow the evaluations of J_{22}, J_{31}, J_{32}, J_{33}, and drag without much loss of accuracy and computational efficiency.

In summary the analytic Kepler+ algorithm includes accelerations due to J_2, J_3, J_4, J_{22}, J_{31}, J_{32}, J_{33}, Sun, Moon and drag wherever applicable. Since most of the prediction errors are eliminated, daily SSA position database updates of the current 20,000 to the future 100,000 or more objects can be achieved with speed, accuracy and robustness. Again, initial TLE errors can either be reduced by the method of Least Squares similar to those of References 33 to 36, or using the more accurate SP Catalog data directly. If prediction and initial errors can be reduced, then a few more days of fast and accurate conjunction and collision assessments become possible. The classified SP data, which has minimal initial errors, is unfortunately not available to the public. The use of SP data and Kepler+, of course, provides the most efficient pre-conjunction results.

Even though the input and output of Vinti and Kepler+ are osculating position and velocity vectors as the SP data of numerical integration, the use of TLE input data with Vinti and Kepler+ is simple and straightforward. That is, the Vinti and Kepler+ algorithms can use either SP or TLE as input data for fast, accurate and robust predictions. However, given the SP data, the SGP4 algorithm cannot be used for predictions, unless the o2m conversions are performed. As stated in the SGP4 difficulties #6 and #7 of the SGP4 section, the converted single precision mean orbital elements, which will introduce additional errors to subsequent SGP4 predictions, make the SP/SGP4 combination practically useless. In other words, SGP4 is made to predict with single precision TLE's as input. The unnatural use of SGP4 with osculating position and velocity vectors of the SP data introduces punishing prediction errors due to additional initial errors from single precision conversion.

In the next section numerical solutions of three classical (2-body) Kepler algorithms are compared in Examples 1 and 2. Examples 3 and 4 compare numerical solutions of 2-body Kepler, SGP4, Vinti, Kepler+, and Numerical Integration (4x4 Geopotentials, Sun, Moon, Drag) using solutions of Numerical Integration (12x12, Sun, Moon, Drag) as reference. Many other days of TLE files were also used to deduce the performance parameters, and the results are similar to those as shown in Tables 1 to 3.

The serious reader can also use the **iSGP** Demo Tool to deduce performance parameters with any other desired TLE files. If your verifications produce unreasonable results by using any other days of TLE files, notice to the author will be gratefully accepted. Your correct results will be published immediately on the *DerAstrodynamics* website to inform interested readers.

Numerical Examples

Kepler Algorithm Examples

Example 1: High Earth Orbit, (2007_249, Satellite ID (SID) = 28836)
kepler1a failed. **kepler1b** and **kepler2** produced good solutions.

Input

$t_1 = 0, \ t_2 = 86{,}400$ seconds (1 day)

r_eci (t_1) (km)			**v_eci** (t_1) (km/s)		
2218.922362	–13049.809380	–15.285102	3.894738598	4.630875574	2.332314347

Output

Algorithm	**r_eci** (t_2) (km)			**v_eci** (t_2) (km/s)		
kepler1a	–53052.0539	–54034.8022	–30994.6668	–6.451059	4.143735	–2.850977
kepler1b	–27456.2663	–16961.1113	–15098.0763	1.431551	–1.341058	0.596545
kepler2	–27456.2663	–16961.1113	–15098.0763	1.431551	–1.341058	0.596545
num. int. (2-body)	–27456.2661	–16961.1114	–15098.0763	1.431551	–1.341058	0.596545

Example 2: High Earth Orbit, 2009_298, SID = 19773
kepler1b failed. **kepler1a** and **kepler2** produced good solutions.

Input

$t_1 = 0, \ t_2 = 86{,}400$ seconds (1 day)

r_eci (t_1) (km)			**v_eci** (t_1) (km/s)		
–209.011468	–30385.859968	4.280644	2.248716032	2.177749649	0.329155365

Output

Algorithm	**r_eci** (t_2) (km)			**v_eci** (t_2) (km/s)		
kepler1a	–16906.0049	1871.5115	–2495.6926	–4.125426	–3.558097	–0.604363
kepler1b	99927.4767	92921.1426	14631.3162	–5.642214	7.607469	–0.840981
kepler2	–16906.0049	1871.5115	–2495.6926	–4.125426	–3.558097	–0.604363
num. int. (2-body)	–16906.0054	1871.5111	–2495.6927	–4.125426	–3.558097	–0.604363

SGP4, **Vinti**, **Kepler**+, and **Numerical Integration Examples** (additional examples can be found in Reference 32)

Example 3: Low Earth Orbit, 2009_249, SID = 4382
kepler2 is poor. **sgp4** is fair. **vinti** is reasonable.
kepler+ and **num. int. (4x4, . .)** produced good solutions for SSA purposes.

Input

$t_1 = 0$, $t_2 = 86{,}400$ seconds (1 day)

r_eci (t_1) (km)			**v_eci** (t_1) (km/s)		
510.306735	6793.878710	0.007287	–2.949065600	0.256823630	7.486987267

Output

Algorithm	**r_eci** (t_2) (km)			**v_eci** (t_2) (km/s)		
kepler2	346.687092	6795.012489	412.914501	–2.978863	–0.215631	7.472634
sgp4	257.913989	6719.684754	1214.516634	–3.040136	–1.016065	7.361291
vinti	207.380964	6701.510662	1337.255798	–3.045055	–1.156391	7.334489
kepler+	213.326456	6703.763322	1322.957867	–3.044553	–1.140038	7.337721
num. int. (4x4, . .)	213.622528	6703.908820	1322.028866	–3.044516	–1.139010	7.337934
num. int. (12x12, . .)	216.109788	6704.762171	1316.193868	–3.044353	–1.132308	7.339295

Example 4: Low Earth Orbit, 2009_249, SID = 23282
kepler2 is poor. **sgp4** is fair. **vinti** is reasonable.
kepler+ and **num. int. (4x4, . .)** produced good solutions for SSA purposes.

Input

$t_1 = 0, \; t_2 = 86{,}400$ seconds (1 day)

r_eci (t_1) (km)			**v_eci (t_1)** (km/s)		
6969.687412	2195.624199	–0.030412	–0.695626778	2.267918155	6.958092777

Output

Algorithm	**r_eci (t_2)** (km)			**v_eci (t_2)** (km/s)		
kepler2	5496.729940	3059.667485	3715.519008	–4.385493	0.712399	5.858270
sgp4	5233.338371	2920.278629	4171.411336	–4.806832	0.629942	5.529725
vinti	5240.654526	2919.268575	4163.061193	–4.798649	0.634552	5.536259
kepler+	5248.036696	2918.287891	4154.564282	–4.790380	0.639186	5.542832
num. int. (4x4, . .)	5248.183301	2918.362041	4154.359585	–4.790136	0.639259	5.543012
num. int. (12x12, . .)	5250.523888	2917.968463	4151.680633	–4.787513	0.640780	5.545121

Conclusions

Analytic classical Kepler algorithms, which are imbedded in all advanced Kepler algorithms, hold the key to the success or failure of analytic Astrodynamics algorithms. Algorithms that solve the Kepler equation in Cartesian coordinates to give directly the position and velocity vectors instead of the Keplerian elements (a, e, i, Ω, ω, M) are much more robust. A classical Kepler algorithm may fail due to any or all of the following: 1. the iterative method (Newton, Halley, Laguerre, . . ,), 2. the initial guess for the iterative method and 3. the correction formula after each iteration. The kepler2 algorithm has yet to fail after more than five billion arbitrary test cases using initial state vectors of the NASA 2015 and 2030 Debris Catalogs.

Three analytic advanced Kepler algorithms are used for trajectory predictions to compare speed, accuracy, robustness, ease of use and implementation simplicity. For more than 50 years, the SGP4 algorithm has been used and assumed as the only analytic prediction computer program for satellites and space debris. The Vinti algorithm and computer source code, which are fast, accurate and robust in general, have been developed and implemented in missile guidance systems since the 1960s in addition to satellite and space debris predictions. Vinti algorithms [32] are available to the public, but often misinterpreted due to their advanced nature and neglected due to the lack of self-promotion. The new Kepler+ algorithm is an extension of Vinti developed to satisfy more stringent SSA requirements.

If accurate SP data are available and computing costs and real-time results are not important, advanced analytic prediction algorithms are not needed for known (cataloged) objects. However, for unknown objects (newly detected UCTs), fast, accurate and robust predictions are invaluable for SSA applications, especially catalog building. Interested readers should perform their independent research to verify results similar to those in Tables 1 to 3 and A1 to A3. To conclude this paper, we revisit the contentions of SGP4 and Vinti:

1. Covered-up: Numerous Astrodynamics papers [22] and textbooks disregarded the existence of the Vinti algorithm, which in turn denied students to learn the one and only advanced application of Hamilton-Jacobi theory for Astrodynamics.
2. Real Purpose: The NRC paper [24] indicates that the SGP4 is mainly for catalog maintenance, while satellite and space debris predictions need improvements.
3. Singularity: Important satellites in the Molniya orbit regime need costly reacquisition for years due to erroneous pointing predictions by SGP4. A version of the Vinti algorithms was developed at NSA [39] indicating that the Agency knew the importance of the singularity-free Vinti algorithm for all satellite and missile predictions.
4. Misinterpretation and Truth: The Woodburn 2005 presentation [37] shows the misunderstanding and ignorance of many experts that Vinti is not a general perturbations theory contradicting GTDS [38] and other research results since 1960s.
5. Blind Leading Blind Theory: General Campbell in Aviation Week 2012 [40] pointed out that SDA does not really provide any conjunction assessment against 20,000 or so pieces of debris that are in the LEO environment. However, AGI was hired by SDA to develop the computer tools needed to make it work, and therefore their conjunction assessments for GEO objects may be alright, but those of LEO objects are really hard to believe.

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Appendix A Prediction performance by orbit regimes including 2-body solutions

The objectives of this Appendix are to find:

1. How many objects have 2-body Kepler and SGP4 solutions with errors greater than 10 km (can be different, if desired) and how many should be replaced.
2. The mean and standard deviation of position differences in the three orbit regimes (LEO, Deep Space - non-GEO, and GEO).

The one-day prediction solutions of 2-body Kepler, SGP4, Kepler+ and numerical integration (WGS 4x4 Geopotentials, Sun, Moon and drag) are collected with respect to LEO, Deep Space (non-GEO) and GEO objects using the three TLE files of Tables 1 to 3. However, all objects in each TLE file are included, since the accuracy of predictions is the main concern.

In Table A1, the total number of objects is 12049 in the TLE file of 6 Sep 2007. The number of LEO, Deep Space (non-GEO) and GEO objects are respectively 9484, 1661 and 904. If the position difference from the reference solution is greater than 10 km, then all 9484 LEO objects with Kepler (2-body) solutions need to be replaced, while only 2463 LEO objects with SGP4 solutions are replaced. The mean and standard deviation (std) for LEO objects are largest as compared with Deep Space (non-GEO) and GEO objects. Results of other two TLE files (25 Oct 2009, 5 Jan 2011) are similar for all objects in all orbit regimes.

The prediction solutions of Deep Space (non-GEO) and GEO objects are illustrated in Tables A2 and A3.

Algorithm (Reference = WGS84 12x12, Sun, Moon, Drag)	Robustness, mean and standard deviation (std) for one-day prediction					
	6 Sep 2007 total # of objs = 12049		25 Oct 2009 total # of objs = 15312		5 Jan 2011 total # of objs = 15929	
	# of objects replaced by iSGP / all	mean / std (km)	# of objects replaced by iSGP / all	mean / std (km)	# of objects replaced by iSGP / all	mean / std (km)
Classical Kepler (2-body)	9484 / 9484	494 / 198	12420 /12421	495 / 199	12944 /12944	494 / 182
SGP4 (analytic)	2463 / 9484	16.5 / 96.	3427 / 12421	17.6 / 141.	3663 / 12944	14.7 / 60.
Kepler+ (analytic)	- - -	6.4 / 8.4	- - -	6.6 / 14.	- - -	6.0 / 8.0
Numerical Integration (4x4, 200s)	- - -	1.8 / 1.2	- - -	1.9 / 1.3	- - -	1.9 / 1.2

Table A1. Robustness, mean and standard deviation comparison of prediction algorithms for one-day prediction on LEO cataloged objects

Algorithm (Reference = WGS84 12x12, Sun, Moon, Drag)	Robustness, mean and standard deviation (std) for one-day prediction					
	6 Sep 2007 total # of objs = 12049		25 Oct 2009 total # of objs = 15312		5 Jan 2011 total # of objs = 15929	
	# of objects replaced by iSGP / all	mean / std (km)	# of objects replaced by iSGP / all	mean / std (km)	# of objects replaced by iSGP / all	mean / std (km)
Classical Kepler (2-body)	1651 / 1661	424 / 695	1899 / 1910	434 / 524	1956 / 1973	398 / 475
SGP4 (analytic)	1288 / 1661	294 / 855	1494 / 1910	307 / 543	1508 / 1973	275 / 469
Kepler+ (analytic)	- - -	21. / 505.	- - -	4.2 / 9.7	- - -	4.6 / 11.
Numerical Integration (4x4, 200s)	- - -	0.7 / 1.4	- - -	0.7 / 1.4	- - -	0.5 / 1.2

Table A2. Robustness, mean and standard deviation comparison of prediction algorithms for one-day prediction on Deep Space (non-GEO) cataloged objects

Algorithm (Reference = WGS84 12x12, Sun, Moon, Drag)	Robustness, mean and standard deviation (std) for one-day prediction					
	6 Sep 2007 total # of objs = 12049		25 Oct 2009 total # of objs = 15312		5 Jan 2011 total # of objs = 15929	
	# of objects replaced by iSGP / all	mean / std (km)	# of objects replaced by iSGP / all	mean / std (km)	# of objects replaced by iSGP / all	mean / std (km)
Classical Kepler (2-body)	708 / 904	15. / 5	791 / 981	15. / 5.	496 / 1012	11. / 8.
SGP4 (analytic)	2 / 904	4.2 / 2.5	1 / 981	3.8 / 2.3	0 / 1012	7. / 4.
Kepler+ (analytic)	- - -	4.2 / 2.5	- - -	3.8 / 2.2	- - -	7. / 4.
Numerical Integration (4x4, 200s)	- - -	0.0 / 0.0	- - -	0.0 / 0.0	- - -	0.0 / 0.0

Table A3. Robustness, mean and standard deviation comparison of prediction algorithms for one-day prediction on GEO cataloged objects

